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PRICE: primitive centred schemes for hyperbolic systems

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SUMMARY

We present first- and higher-order non-oscillatory primitive (PRI) centred (CE) numerical schemes for solving systems of hyperbolic partial differential equations written in primitive (or non-conservative) form. Non-conservative systems arise in a variety of fields of application and they are adopted in that form for numerical convenience, or more importantly, because they do not posses a known conservative form; in the latter case there is no option but to apply non-conservative methods. In addition we have chosen a *centred*, as distinct from *upwind*, philosophy. This is because the systems we are ultimately interested in (e.g. mud flows, multiphase flows) are exceedingly complicated and the eigenstructure is difficult, or very costly or simply impossible to obtain. We derive six new basic schemes and then we study two ways of extending the most successful of these to produce second-order non-oscillatory methods. We have used the MUSCL-Hancock and the ADER approaches. In the ADER approach we have used two ways of dealing with linear reconstructions so as to avoid spurious oscillations: the ADER TVD scheme and ADER with ENO reconstruction. Extensive numerical experiments suggest that all the schemes are very satisfactory, with the ADER/ENO scheme being perhaps the most promising, first for dealing with source terms and secondly, because higher-order extensions (greater than two) are possible. Work currently in progress includes the application of some of these ideas to solve the mud flow equations. The schemes presented are generic and can be applied to any hyperbolic system in non-conservative form and for which solutions include smooth parts, contact discontinuities and weak shocks. The advantage of the schemes presented over upwind-based methods is simplicity and efficiency, and will be fully realized for hyperbolic systems in which the provision of upwind information is very costly or is not available. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: hyperbolic systems; primitive form; primitive methods; centred methods

1. INTRODUCTION

This paper is about designing numerical methods of the centred type (non-upwind) for computing solutions to hyperbolic equations in primitive (non-conservative) form. In the last two to three decades, there has been tremendous progress in the development and application of

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good numerical methods for hyperbolic conservation laws. Up-to-date background information is found, for example, in the textbooks [1–5]. Almost universally, these methods are designed to solve the equations in conservative form, and the most accurate schemes use explicit wave propagation information (upwinding) in the schemes. The theme of this paper is somewhat unconventional, in that we wish to design non-conservative (or primitive) methods. The reason we want to use primitive schemes is that there are systems of hyperbolic equations that have no known conservative form. This may appear paradoxical to the reader, in that although the equations may have been derived from physical conservation principles, yet the resulting equations cannot be expressed in conservation-law form, or divergence form. Prominent examples arise in the modelling of mud flows [6–8]. Multiphase flow modelling is another important area of application in which the resulting governing equations cannot always be expressed in conservation-law form [9, 10]. As a matter of fact, in the nuclear and propulsion industries for example, there is currently a lot of research activity aimed at deriving multiphase flow models that can be expressed in conservative form. The reason for searching for conservative models is that the generalization of classical (smooth) solutions of the equations to include shock waves (weak solutions) is straightforward. This is not so for non-conservative systems, although some interesting work in this direction is already available [11, 12], whereby non-conservative products may be defined at the cost of having to provide extra information. However, it is unclear to these authors how such work would be applicable to the physical situations mentioned above. Thus in order to solve existing non-conservative systems numerically one must design appropriate (non-conservative) numerical methods. When solutions are smooth there is no good reason known to these authors why one should not use a non-conservative (primitive) method. In practice these methods perform very well [13] and in fact there are situations in which primitive methods outperform conservative methods [14]. Linear discontinuities (such as contact and shear waves) are well-represented by primitive methods, those that are non-oscillatory of course. Shock waves will, however, be computed with the wrong strength and the wrong propagation speed. In fact there is a theorem due to Hou and LeFloch [15] in which it is proved that non-conservative methods will converge to the wrong solution in the presence of a shock wave. For weak shocks, however, the methods put forward in this paper would be adequate.

The second aspect of this work concerns the philosophy to be followed in designing the schemes. Primitive (non-conservative) schemes using upwinding have been put forward in the past. Examples include the works of Karni [16] and Toro [17, 13]. Upwind schemes require the explicit provision of wave propagation information. Normally this is achieved via local solutions of the Riemann problem, approximate or exact. For most known hyperbolic systems, the exact or approximate solution of the Riemann problem is available, although in some cases this may be very expensive to evaluate. The systems we are interested in are very complicated and so far the solution of the Riemann problem is not, to our knowledge, available. We therefore have decided to adopt a *centred* approach, in which no explicit information regarding wave propagation is used in the scheme, apart from stability constraints via a Courant (or CFL) condition, for which at least the eigenvalues of the system must be known, even if it is only numerically. But this information must in any case be available, as knowledge on the nature of the eigenvalues would inform us on the character of the equations being solved, hyperbolic or otherwise.

In this paper we develop primitive (PRI) centred (CE) schemes for non-conservative hyperbolic systems in one and multiple space dimensions. Our PRICE schemes will be based

on the design of first-order, monotone, primitive centred schemes. These schemes are then extended to second order of accuracy following two approaches, namely the MUSCL-Hancock approach [18, 3] and the ADER approach [19–21]. In the MUSCL-Hancock approach spurious oscillations are controlled via a Total Variation Diminishing (TVD) constraint. In the ADER approach we use two types of non-linear (non-oscillatory) extensions, namely (i) a TVD approach similar to the MUSCL-Hancock scheme, and (ii) least oscillatory polynomial reconstructions of the ENO type [22, 21]. We also carry out an extensive and systematic assessment of the designed schemes by solving the non-linear shallow water equations, written in primitive form, augmented by a passive scalar associated with the modelling of sediment transport. The assessment is aimed at identifying the best schemes to be recommended for practical use, as what we have in mind next is the application of the most successful schemes to the solution of non-conservative equations arising in the modelling of mud flows [8].

In Section 2 we design six basic primitive centred schemes. In Section 3 we construct second-order non-oscillatory extensions of the most promising first-order scheme developed in Section 2. In Section 4 we extensively and systematically assess the methods by solving the (augmented) one-dimensional non-linear shallow water equations and compare results with exact solutions. Conclusions are drawn in Section 5. In Appendix A we discuss TVD constraints and slope limiters.

2. DESIGNING PRIMITIVE CENTRED SCHEMES

2.1. Introduction

We are interested in solving hyperbolic partial differential equations of the form:

$$\partial_t \mathbf{Q} + \mathbf{A}(\mathbf{Q})\partial_x \mathbf{Q} = \mathbf{0} \tag{1}$$

in which $\mathbf{Q} = [q_1, \ldots, q_n]^{\mathrm{T}}$ is the vector of unknowns and $\mathbf{A} = \mathbf{A}(\mathbf{Q})$ is the coefficient matrix. We assume system (1) to be hyperbolic with real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and a set of corresponding linearly independent right eigenvectors $\mathbf{R^{(1)}}, \ldots, \mathbf{R^{(n)}}$. We note here that the numerical methods developed in this paper only require an estimate for the maximum of the eigenvalues in absolute value; this is in order to enforce a stability condition.

The vector of unknowns $\mathbf{Q}$ in (1) could be the vector of physically conserved variables, in which case $\mathbf{A}(\mathbf{Q})$ would the Jacobian matrix $\mathbf{A}(\mathbf{Q}) \equiv \partial \mathbf{F}/\partial \mathbf{Q}$ and $\mathbf{F} = \mathbf{F}(\mathbf{Q})$ would be the physical flux, a vector-valued function. In such case (1) could be expressed in conservative form

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \tag{2}$$

and one could then apply any of the modern conservative shock-capturing methods available [1–4] to solve Equation (2) numerically.

There are cases in which a non-conservative formulation (1) has some advantages over the conservative formulation (2). More importantly, there are many situations of practical interest in which no known conservative form of the equations exists. This is the case in environmental fluid dynamics, two-phase flows and other application areas, and this is in fact the main motivation of this paper.

2.2. PRICE schemes of the Lax–Wendroff–Godunov type

2.2.1. The framework and updating formula. Consider a control volume $V_i = [x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]$ in $x-t$ space. Integration of (1) in V_i gives

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t^n}^{t^{n+1}} [\partial_t \mathbf{Q} + \mathbf{A}(\mathbf{Q})\partial_x \mathbf{Q}] \, \mathrm{d}x \, \mathrm{d}t = \mathbf{0} \tag{3}$$

Assume a local linearization in (3) with a constant matrix $\hat{\mathbf{A}}_i$. Then, integration gives

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}(x, t^{n+1}) \, \mathrm{d}x = \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}(x, t^n) \, \mathrm{d}x - \hat{\mathbf{A}}_i \left[\int_{t^n}^{t^{n+1}} \mathbf{Q}(x_{i+1/2}, t) \, \mathrm{d}t - \int_{t^n}^{t^{n+1}} \mathbf{Q}(x_{i-1/2}, t) \, \mathrm{d}t \right] \tag{4}$$

Define space and time averages as

$$\mathbf{Q}_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}(x, t^n) \, \mathrm{d}x, \quad \mathbf{Q}_{i+1/2} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{Q}(x_{i+1/2}, t) \, \mathrm{d}t \tag{5}$$

where $\Delta x = x_{i+1/2} - x_{i-1/2}$ and $\Delta t = t^{n+1} - t^n$. With these definitions (4) becomes

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x} \hat{\mathbf{A}}_i [\mathbf{Q}_{i+1/2} - \mathbf{Q}_{i-1/2}] \tag{6}$$

Assuming the control volume V_i defines a mesh in $x-t$ space, then (6) provides a numerical formula to advance the solution in time from time level n to time level $n+1$, provided the coefficient matrix $\hat{\mathbf{A}}_i$ and the *intermediate state* $\mathbf{Q}_{i+1/2}$ are defined, which now is an approximation of the corresponding time average in (5).

As to the choice of the coefficient matrix $\hat{\mathbf{A}}_i$ one could, for example, select

$$\hat{\mathbf{A}}_i = \mathbf{A}(\mathbf{Q}_i^n) \equiv \mathbf{A}_i^n \tag{7}$$

Other choices are also possible. Regarding the choice of *intermediate states* $\mathbf{Q}_{i+1/2}$ we shall adopt a centred (non-upwind) approach. Upwind non-conservative schemes were put forward in Reference [13].

Remark

A more elaborate framework results from keeping the matrix $\mathbf{A}(\mathbf{Q})$ in (3) and using integration by parts. This lead to

$$\left.\begin{aligned} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}(x, t^{n+1}) \, \mathrm{d}x = \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}(x, t^n) \, \mathrm{d}x - \int_{t^n}^{t^{n+1}} [\mathbf{A}(\mathbf{Q})\mathbf{Q}(x_{i+1/2}, t) \\ - \mathbf{A}(\mathbf{Q})\mathbf{Q}(x_{i-1/2}, t)] \mathrm{d}t - \int_{t^n}^{t^{n+1}} \left[\int_{x_{i-1/2}}^{x_{i+1/2}} \partial_x \mathbf{A}(\mathbf{Q})\mathbf{Q} \, \mathrm{d}x \right] \mathrm{d}t \end{aligned}\right\} \tag{8}$$

This would result in more complex schemes and we have not pursued this route further, as the simpler schemes (6) lead to very satisfactory results.

Multidimensional versions of (6) that are *unsplit* or *simultaneous updating* formulae can derived in a similar way. For example, for a two-dimensional system in Cartesian co-ordinates

$$\partial_t \mathbf{Q} + \mathbf{A}(\mathbf{Q})\partial_x \mathbf{Q} + \mathbf{B}(\mathbf{Q})\partial_y \mathbf{Q} = 0 \tag{9}$$

one can write schemes of the form

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{ij}[\mathbf{Q}_{i+1/2,j} - \mathbf{Q}_{i-1/2,j}] - \frac{\Delta t}{\Delta y}\hat{\mathbf{B}}_{ij}[\mathbf{Q}_{i,j+1/2} - \mathbf{Q}_{i,j-1/2}] \tag{10}$$

Similar formulae may be derived for the 3D case. Alternatively, one may use dimensional splitting to extend (6) to solve (9) and its 3D version.

In the following subsections we construct *intermediate states* $\mathbf{Q}_{i+1/2}$ that completely determine the solution updating schemes (6) and (10). We shall study two approaches, both of them using centred (non-upwind) schemes.

2.2.2. Some useful integral relations. The task at hand now is to obtain expressions for the intermediate vectors $\mathbf{Q}_{i+1/2}$, that are some kind of numerical analogue of the time average given in (5). We obtain an expression for $\mathbf{Q}(x_{i+1/2}, \alpha\Delta t)$, where $0 \leqslant \alpha \leqslant 1$, by integrating (1) in the control volume $[x_{i+1/2} - \frac{1}{2}\Delta x, x_{i+1/2} + \frac{1}{2}\Delta x] \times [0, \alpha\Delta t]$ following an analogous procedure to the one leading to (6), namely

$$\begin{aligned}
&\int_{x_{i+1/2}-\frac{1}{2}\Delta x}^{x_{i+1/2}+\frac{1}{2}\Delta x} \mathbf{Q}(x, \alpha\Delta t)\,\mathrm{d}x \\
&\quad = \int_{x_{i+1/2}-\frac{1}{2}\Delta x}^{x_{i+1/2}+\frac{1}{2}\Delta x} \mathbf{Q}(x, 0)\,\mathrm{d}x - \hat{\mathbf{A}}_{i+1/2} \\
&\qquad \times \left[\int_0^{\alpha\Delta t} \mathbf{Q}\left(x_{i+1/2} + \frac{1}{2}\Delta x, t\right)\mathrm{d}t - \int_0^{\alpha\Delta t} \mathbf{Q}\left(x_{i+1/2} - \frac{1}{2}\Delta x, t\right)\mathrm{d}t\right]
\end{aligned} \tag{11}$$

Dividing through by Δx and setting

$$\mathbf{Q}\left(x_{i+1/2} - \frac{1}{2}\Delta x, t\right) = \mathbf{Q}_i^n, \quad \mathbf{Q}\left(x_{i+1/2} + \frac{1}{2}\Delta x, t\right) = \mathbf{Q}_{i+1}^n, \quad \hat{\mathbf{A}}_{i+1/2} = \mathbf{A}\left(\frac{1}{2}[\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n]\right) \tag{12}$$

under a CFL-like condition, we obtain

$$\mathbf{Q}_{i+1/2}^{\alpha} = \frac{1}{\Delta x}\int_{x_{i+1/2}-\frac{1}{2}\Delta x}^{x_{i+1/2}+\frac{1}{2}\Delta x} \mathbf{Q}(x, \alpha\Delta t)\,\mathrm{d}x = \frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \alpha\frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{i+1/2}(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n) \tag{13}$$

We can now construct schemes (6) by direct use of integral relation (13) to obtain intermediate states.

2.2.3. PRICE-LW: a primitive Lax–Wendroff scheme. For $\alpha = \frac{1}{2}$ in (13) we obtain

$$\mathbf{Q}_{i+1/2}^{\text{PRICE-LW}} = \frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{i+1/2}(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n) \tag{14}$$

Inserting this into the solution updating formula (6) gives a scheme that is analogous to the two-step Lax–Wendroff method developed originally for conservative systems [23]. The resulting PRICE-LW scheme is second-order accurate in space and time and oscillatory in the vicinity of large gradients.

2.2.4. PRICE-G: the Godunov centred approach. Godunov [24] developed a centred *conservative* scheme (not to be confused with his well-known upwind scheme) that is not particularly well-known but has some attractive and intriguing features. Here we develop a non-conservative analogue to be applied to (1) via (6). In fact the way the intermediate state $\mathbf{Q}_{i+1/2}$ is found is entirely analogous to that leading to (6), (13), in which we simply take $\alpha = 1$ leading to

$$\mathbf{Q}_{i+1/2}^{\text{PRICE-G}} = \frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{i+1/2}(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n) \tag{15}$$

with $\hat{\mathbf{A}}_{i+1/2}$ as in (12). Inserting this into the solution updating formula (6) gives the primitive centred Godunov scheme PRICE-G, which when applied to the model equation

$$\partial_t q + \lambda \partial_x q = 0 \tag{16}$$

with λ a constant wave propagation speed, gives

$$q_i^{n+1} = \frac{1}{2}c(1+2c)q_{i-1}^n + (1-2c^2)q_i^n - \frac{1}{2}c(1-2c)q_{i+1}^n \tag{17}$$

Here $c = \lambda(\Delta t/\Delta x)$ is the Courant number, or CFL number. It is easy to show that this scheme has linearized stability condition

$$0 \leqslant |c| \leqslant \frac{1}{2}\sqrt{2} \tag{18}$$

It is interesting to note that this first-order scheme is *not monotone* in the stability range $0 \leqslant |c| \leqslant \frac{1}{2}$ and is monotone in the stability range $\frac{1}{2} \leqslant |c| \leqslant \frac{1}{2}\sqrt{2}$. Our experience is that the loss of monotonicity is not too serious for this scheme, in the sense that the spurious oscillations produced near large gradients have small amplitude. Strictly speaking, however, any attempt at constructing high-order extension of the TVD type, for example, will be unsuccessful. These schemes can be constructed under the assumption that the underlying first-order scheme is monotone throughout its stability range.

2.3. PRICE-LF: a Lax–Friedrichs type centred scheme

The point of departure here is the observation [3] that the conservative Lax–Friedrichs scheme can be constructed from

$$\mathbf{Q}_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \hat{\mathbf{Q}}_i\left(x, \frac{1}{2}\Delta t\right) \mathrm{d}t \tag{19}$$

when $\mathbf{Q}_i$ represents the vector of conserved variables and $\hat{\mathbf{Q}}_i(x,t)$ is the solution of the Riemann problem for the relevant system of conservation laws with initial condition

$$\mathbf{Q}(x,0)=\begin{cases}\mathbf{Q}_{i-1}^n & \text{if } x<0\\ \mathbf{Q}_{i+1}^n & \text{if } x>0\end{cases} \tag{20}$$

We emphasize that the solution $\hat{\mathbf{Q}}_i$ has as its initial states the two neighbours at $i-1$ (left) and $i+1$ (right). See Section 7.3.1 of Reference [3] for details on this interpretation of the Lax–Friedrichs scheme for conservative systems.

Here we apply the same approach to the non-conservative system (1) and in which $\mathbf{Q}$ in (19) is any set of representative variables, including conservative variables. Following steps analogous to those leading to the derivation of (13) we obtain

$$\mathbf{Q}_i^{n+1}=\frac{1}{2}\left(\mathbf{Q}_{i-1}^n+\mathbf{Q}_{i+1}^n\right)-\frac{1}{2}\frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_i\left(\mathbf{Q}_{i+1}^n-\mathbf{Q}_{i-1}^n\right) \tag{21}$$

with the matrix $\hat{\mathbf{A}}_i$ evaluated as

$$\hat{\mathbf{A}}_i=\mathbf{A}\left(\frac{1}{2}[\mathbf{Q}_{i+1}^n+\mathbf{Q}_{i-1}^n]\right) \tag{22}$$

This scheme may be rewritten as

$$\mathbf{Q}_i^{n+1}=\mathbf{Q}_i^n-\frac{\Delta t}{\Delta x}[\mathbf{Q}_{i+1/2}^L-\mathbf{Q}_{i-1/2}^R] \tag{23}$$

with intermediate states

$$\mathbf{Q}_{i+1/2}^L=\frac{1}{2}\hat{\mathbf{A}}_i(\mathbf{Q}_i^n+\mathbf{Q}_{i+1}^n)-\frac{1}{2}\frac{\Delta x}{\Delta t}(\mathbf{Q}_{i+1}^n-\mathbf{Q}_i^n) \tag{24}$$

$$\mathbf{Q}_{i-1/2}^R=\frac{1}{2}\hat{\mathbf{A}}_i(\mathbf{Q}_{i-1}^n+\mathbf{Q}_i^n)-\frac{1}{2}\frac{\Delta x}{\Delta t}(\mathbf{Q}_i^n-\mathbf{Q}_{i-1}^n) \tag{25}$$

It is easy to check that the above two different forms (21) and (23) of the PRICE-LF scheme are equivalent. The scheme that we have obtained, when applied to the model equation (16) gives the conventional Lax–Friedrichs scheme. This scheme is first-order accurate, monotone and with the stability condition

$$|c|\leqslant 1 \tag{26}$$

where c is the CFL number.

2.4. PRICE schemes of the FORCE type

Here we construct primitive schemes that are analogues of the FORCE method, first developed for conservative systems. Anile *et al.* [25] appear to be the first authors to have constructed a primitive analogue of the FORCE scheme. Their approach, however, requires the use of the

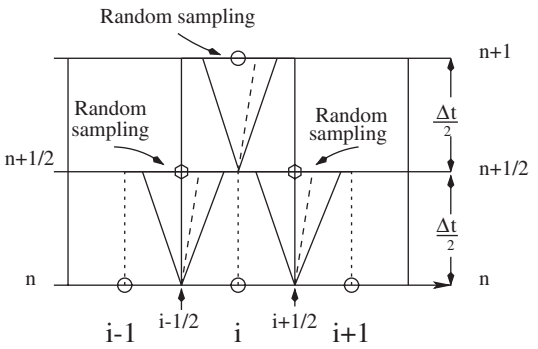

Figure 1. Illustration of the random choice method (RCM) on a staggered grid.

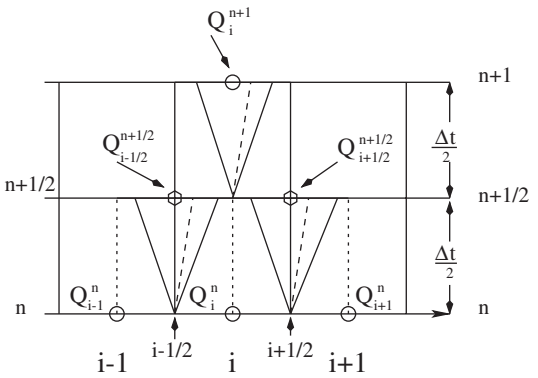

Figure 2. Illustration of the PRICE-T scheme derived from the staggered-grid RCM.

conservative form of the equations, whereas the schemes presented here do not and are thus more general.

2.4.1. PRICE-T: a scheme based on staggered grid RCM. A deterministic re-interpretation [26] of the staggered-grid version of the Random Choice Method (RCM) of Glimm [27], whereby randomly sampled solutions of local Riemann problems are replaced by integral averages, leads to a first-order centred conservative method, called the FORCE method [26]; see also References [3, 28]. In the conservative case, it is surprising to find that the resulting numerical flux is the arithmetic mean of the Lax–Friedrichs flux and the two-step Lax–Wendroff flux. Here we construct an analogous non-conservative scheme that has a form different from that of the conservative FORCE scheme. This is shown below.

We first review of RCM on a staggered grid. The staggered grid version of the RCM updates $\mathbf{Q}_i^n$ to a new value $\mathbf{Q}_i^{n+1}$ in two steps, as illustrated in Figures 1 and 2. The steps are:

- **Step (I).** Solve the Riemann problems RP($\mathbf{Q}_{i-1}^n, \mathbf{Q}_i^n$) and RP($\mathbf{Q}_i^n, \mathbf{Q}_{i+1}^n$) to find respective solutions

$$\hat{\mathbf{Q}}_{i-1/2}^{n+1/2}(x,t), \quad \hat{\mathbf{Q}}_{i+1/2}^{n+1/2}(x,t) \tag{27}$$

Random sample these solutions at a stable time $\frac{1}{2}\Delta t$, that is

$$\mathbf{Q}_{i-1/2}^{n+1/2} = \hat{\mathbf{Q}}_{i-1/2}^{n+1/2}(\theta^n \Delta x, \Delta t^{n+1/2}), \quad \mathbf{Q}_{i+1/2}^{n+1/2} = \hat{\mathbf{Q}}_{i+1/2}^{n+1/2}(\theta^n \Delta x, \Delta t^{n+1/2}) \tag{28}$$

where θ^n is a random number in the interval $[-\frac{1}{2}, \frac{1}{2}]$.

- **Step (II).** Solve RP($\mathbf{Q}_{i-1/2}^{n+1/2}, \mathbf{Q}_{i+1/2}^{n+1/2}$) to find solution $\hat{\mathbf{Q}}_i^{n+1}(x,t)$ and random sample it, at a stable time $\frac{1}{2}\Delta t$, to obtain $\mathbf{Q}_i^{n+1}$, that is

$$\mathbf{Q}_i^{n+1} = \hat{\mathbf{Q}}_i^{n+1}(\theta^{n+1} \Delta x, \Delta t) \tag{29}$$

We now derive the PRICE-T scheme. This is a primitive centred scheme that is obtained by replacing the stochastic steps (28)–(29) by deterministic versions, via integral averages of Riemann problem solutions and using the non-conservative form of Equation (1). The stochastic quantities (28) are replaced by the deterministic integrals

$$\mathbf{Q}_{i-1/2}^{n+1/2} = \frac{1}{\Delta x} \int_{-1/2\Delta x}^{1/2\Delta x} \hat{\mathbf{Q}}_{i-1/2}^{n+1/2}\left(x, \frac{\Delta t}{2}\right) \mathrm{d}x \tag{30}$$

and

$$\mathbf{Q}_{i+1/2}^{n+1/2} = \frac{1}{\Delta x} \int_{-1/2\Delta x}^{1/2\Delta x} \hat{\mathbf{Q}}_{i+1/2}^{n+1/2}\left(x, \frac{\Delta t}{2}\right) \mathrm{d}x \tag{31}$$

Then we apply the integral form of the non-conservative equations (1), namely Equation (4) but on a control volume $V_i = [x_{i+1/2} - \frac{1}{2}\Delta x, x_{i+1/2} + \frac{1}{2}\Delta x] \times [0, \frac{1}{2}\Delta t]$ around $x = x_{i+1/2}$. The result is

$$\mathbf{Q}_{i+1/2}^{n+1/2} = \frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x} \hat{\mathbf{A}}_{i+1/2}[\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n] \tag{32}$$

Similarly,

$$\mathbf{Q}_{i-1/2}^{n+1/2} = \frac{1}{2}(\mathbf{Q}_{i-1}^n + \mathbf{Q}_i^n) - \frac{1}{2}\frac{\Delta t}{\Delta x} \hat{\mathbf{A}}_{i-1/2}[\mathbf{Q}_i^n - \mathbf{Q}_{i-1}^n] \tag{33}$$

We denote by $\hat{\mathbf{Q}}_i(x,t)$ the solution of the Riemann problem RP($\mathbf{Q}_{i-1/2}^{n+1/2}, \mathbf{Q}_{i+1/2}^{n+1/2}$) and define an average $\mathbf{Q}_i^{n+1}$ at the complete time step Δt in terms of an integral average of $\hat{\mathbf{Q}}_i(x,t)$ at the (local) time $t = \frac{1}{2}\Delta t$, namely

$$\mathbf{Q}_i^{n+1} = \frac{1}{\Delta x} \int_{-1/2\Delta x}^{1/2\Delta x} \hat{\mathbf{Q}}_i\left(x, \frac{1}{2}\Delta t\right) \mathrm{d}x \tag{34}$$

This is the deterministic version of (29), which from (13) gives a new solution updating formula

$$\mathbf{Q}_i^{n+1} = \frac{1}{2}(\mathbf{Q}_{i-1/2}^{n+1/2} + \mathbf{Q}_{i+1/2}^{n+1/2}) - \frac{1}{2}\frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_i(\mathbf{Q}_{i+1/2}^{n+1/2} - \mathbf{Q}_{i-1/2}^{n+1/2}) \tag{35}$$

We take

$$\hat{\mathbf{A}}_i = \mathbf{A}\left(\frac{1}{2}[\mathbf{Q}_{i-1/2}^{n+1/2} + \mathbf{Q}_{i+1/2}^{n+1/2}]\right), \quad \hat{\mathbf{A}}_{i+1/2} = \mathbf{A}\left(\frac{1}{2}[\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n]\right) \tag{36}$$

We call PRICE-T the scheme (35) with intermediate states (32). When applied to the model equation (16), the derived PRICE-T scheme is found to be

$$q_i^{n+1} = \frac{1}{4}(1+c)^2 q_{i-1}^n + \frac{1}{2}(1-c^2) q_i^n + \frac{1}{4}(1-c)^2 q_{i+1}^n \tag{37}$$

This is the FORCE scheme [26, 28], which is first-order accurate, monotone and has linearized stability condition (26). As a matter of fact, it has been proved that FORCE is the optimal centred scheme in the sense that it is the least dissipative of all three-point centred methods that are monotone and have stability condition (26); see Reference [29] for details.

2.4.2. PRICE-F. For conservative systems a first-order centred scheme (FORCE) is presented in Reference [26] that is based on the staggered grid version of RCM, just as PRICE-T for non-conservative systems. However, in the case of conservative systems it is fortuitous that the numerical flux turns out to be identically an arithmetic mean of the fluxes for the two-step Lax–Wendroff and Lax–Friedrichs methods. For non-conservative non-linear systems this is not the case, as is shown in the derivation of the PRICE-T scheme. However if we compute an arithmetic average of the solutions obtained from the PRICE-LW and PRICE-LF methods we obtain a non-conservative scheme that is analogous to the FORCE method. In this way, a scheme which we call PRICE-F, is obtained as follows:

$$\mathbf{Q}_i^{n+1(\mathrm{PRICE-F})} = \frac{1}{2}(\mathbf{Q}_i^{n+1(\mathrm{PRICE-LW})} + \mathbf{Q}_i^{n+1(\mathrm{PRICE-LF})}) \tag{38}$$

Here $\mathbf{Q}_i^{n+1(\mathrm{PRICE-LW})}$ is the solution at time level $n+1$ obtained from the PRICE-LW scheme (6), (14) and $\mathbf{Q}_i^{n+1(\mathrm{PRICE\text{-}LF})}$ is the solution at time level $n+1$ obtained from the PRICE-LF scheme (21).

2.4.3. PRICE-S. It is also possible to obtain a non-conservative scheme that is analogous to the FORCE method, by averaging the two intermediate states from the PRICE-LW and the PRICE-LF as follows:

$$\mathbf{Q}_{i+1/2}^{(\mathrm{PRICE-S})} = \frac{1}{2}(\mathbf{Q}_{i+1/2}^{(\mathrm{PRICE-LW})} + \mathbf{Q}_{i+1/2}^{(\mathrm{PRICE-LF})}) \tag{39}$$

where

$$\mathbf{Q}_{i+1/2}^{\mathrm{PRICE-LW}} = \left[\frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{i+1/2}(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n)\right]\mathbf{A}_i \tag{40}$$

and $\mathbf{Q}_{i+1/2}^{(\mathrm{PRICE-LF})}$ is computed as in (24). Inserting (39) into the updating formula

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}[\mathbf{Q}_{i+1/2} - \mathbf{Q}_{i-1/2}] \tag{41}$$

we obtain a new scheme which we call PRICE-S and which is analogous to the FORCE scheme developed originally for conservative systems.

Table I. Basic schemes constructed.

	Basic schemes
PRICE-LW	Primitive centred version of the Lax–Wendroff method
PRICE-G	Primitive centred version of Godunov's method
PRICE-LF	Primitive centred version of the Lax–Friedrichs method
PRICE-T	Primitive centred analogue 1 of the FORCE method
PRICE-F	Primitive centred analogue 2 of the FORCE method
PRICE-S	Primitive centred analogue 3 of the FORCE method

Table II. Summary of intermediate states and updating formulae for the basic schemes constructed.

Basic schemes	Intermediate state $\mathbf{Q}_{i+1/2} =$	Updating formula $\mathbf{Q}_i^{n+1} =$
PRICE-LW	$\frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{i+1/2}(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n)$	$\mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_i[\mathbf{Q}_{i+1/2} - \mathbf{Q}_{i-1/2}]$
PRICE-G	$\frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{i+1/2}(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n)$	$\mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_i[\mathbf{Q}_{i+1/2} - \mathbf{Q}_{i-1/2}]$
PRICE-LF	$[\frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{i+1/2}(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n)]\mathbf{A}_i$	$\mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}[\mathbf{Q}_{i+1/2} - \mathbf{Q}_{i-1/2}]$
PRICE-T	$\frac{1}{2}(\mathbf{Q}_i^n + \mathbf{Q}_{i+1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}\hat{\mathbf{A}}_{i+1/2}(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n)$	$\frac{1}{2}[\mathbf{Q}_{i-1/2}^{n+1/2} + \mathbf{Q}_{i+1/2}^{n+1/2}] - \frac{1}{2}\frac{\Delta t}{\Delta x} \times \hat{\mathbf{A}}_i[\mathbf{Q}_{i+1/2}^{n+1/2} - \mathbf{Q}_{i-1/2}^{n+1/2}]$
PRICE-F	Not needed	$\frac{1}{2}(\mathbf{Q}_i^{n+1(\text{PRICE-LW})} + \mathbf{Q}_i^{n+1(\text{PRICE-LF})})$
PRICE-S	$\frac{1}{2}(\mathbf{Q}_{i+1/2}^{(\text{PRICE}}\text{-LW}) + \mathbf{Q}_{i+1/2}^{(\text{PRICE}}\text{-LF}))$	$\mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}[\mathbf{Q}_{i+1/2} - \mathbf{Q}_{i-1/2}]$

Remark

In general, for a non-linear system we have

$$\mathbf{Q}_i^{n+1(\text{PRICE}-\text{F})} \neq \mathbf{Q}_i^{n+1(\text{PRICE}-\text{T})} \tag{42}$$

$$\mathbf{Q}_i^{n+1(\text{PRICE}-\text{S})} \neq \mathbf{Q}_i^{n+1(\text{PRICE}-\text{T})} \tag{43}$$

$$\mathbf{Q}_i^{n+1(\text{PRICE}-\text{F})} \neq \mathbf{Q}_i^{n+1(\text{PRICE}-\text{S})} \tag{44}$$

where $\mathbf{Q}_i^{n+1(\text{PRICE}-\text{T})}$ is the solution provided by the PRICE−T scheme (35) and (32). Equality is possible for some special ways of linearizing the equations in order to evaluate the coefficient matrix. However for linear systems with constant coefficients equality in (42) holds, and for the model equation (16) all the three schemes reproduce the FORCE method (37), which is first order, monotone and has linearized stability condition (26).

2.5. Summary of the basic schemes

To help the reader we have summarized the main steps of the schemes designed in this section in Tables I and II. Table I describes the schemes, while Table II gives, in the third

column, the updating formula to advance the solution from time level n to time level $n+1$ and the second column gives the intermediate state to advance the solution.

In the next section we study two approaches for constructing second-order non-oscillatory extensions of the first-order schemes studied here.

3. SECOND-ORDER NON-OSCILLATORY EXTENSIONS

We study two approaches. The first follows the MUSCL-Hancock idea originally developed for conservative methods [18] and later applied to non-conservative upwind methods [13]. For background reading see Reference [3, *Section* 14.4]. The second approach follows the advection-diffusion-reaction (ADER) philosophy [19], used to construct non-oscillatory schemes of very high order of accuracy and has so far only been used to design conservative schemes [20, 21]. Both of these approaches for constructing non-oscillatory high order methods require a monotone scheme as the building block. Of the six basic schemes designed in Section 2 there are four schemes that fulfil the monotonicity requirement, namely PRICE-LF, PRICE-T, PRICE-F and PRICE-S. In this section we only implement second-order non-oscillatory extension of the most promising of the basic schemes, namely PRICE-T.

3.1. *MUSCL-Hancock type TVD schemes*

For primitive centred schemes the MUSCL-Hancock approach has the following three steps:

- (I) *MUSCL reconstruction* constrained by a TVD condition, namely

$$\mathbf{Q}_i(x) = \mathbf{Q}_i^n + \frac{(x - x_i)}{\Delta x}\,\bar{\Delta}_i \tag{45}$$

  where $\bar{\Delta}_i$ is a limited slope (difference) so as to avoid spurious oscillations near large gradients of the solution. Details on the evaluation of $\bar{\Delta}_i$ using TVD constraints and slope limiters are given in Appendix A. Boundary extrapolated values of the primitive states are

$$\mathbf{Q}_i^L = \mathbf{Q}_i^n - \frac{1}{2}\,\bar{\Delta}_i, \quad \mathbf{Q}_i^R = \mathbf{Q}_i^n + \frac{1}{2}\,\bar{\Delta}_i \tag{46}$$

- (II) *Evolution of these primitive states* by a time $\frac{1}{2}\Delta t$ as follows:

$$\bar{\mathbf{Q}}_i^{L,R} = \mathbf{Q}_i^{L,R} - \frac{1}{2}\,\frac{\Delta t}{\Delta x}\,\mathbf{A}_i^n(\mathbf{Q}_i^R - \mathbf{Q}_i^L) \tag{47}$$

- (III) *Computation of intermediate states* $\mathbf{Q}_{i+1/2}$. This is done by evaluation of intermediate states applying the PRICE-T scheme of the previous section, namely

$$\mathbf{Q}_{i+1/2}^{\text{PRICE-T}} = \mathbf{Q}_{i+1/2}^{\text{PRICE-T}}(\bar{\mathbf{Q}}_i^R, \bar{\mathbf{Q}}_{i+1}^L) \tag{48}$$

Then the intermediate state becomes

$$\mathbf{Q}_{i+1/2}^{n+1/2} = \frac{1}{2}(\bar{\mathbf{Q}}_i^R + \bar{\mathbf{Q}}_{i+1}^L) - \frac{1}{2}\,\frac{\Delta t}{\Delta x}\,\hat{\mathbf{A}}_{i+1/2}[\bar{\mathbf{Q}}_{i+1}^L - \bar{\mathbf{Q}}_i^R] \tag{49}$$

with

$$\hat{\mathbf{A}}_{i+1/2} = \mathbf{A}\left(\frac{1}{2}(\bar{\mathbf{Q}}_i^R + \bar{\mathbf{Q}}_{i+1}^L)\right) \tag{50}$$

which is then used in the solution updating formula (35).

3.2. *ADER-type schemes*

In this section we develop primitive centred schemes following the ADER philosophy [19]. The ADER approach is a generalization of the Modified GRP scheme of Toro [30], which in turn is based on the second-order GRP approach of Ben-Artzi and Falcovitz [30]. The ADER approach allows the construction of schemes of arbitrary accuracy in both space and time [21, 31, 32].

3.2.1. TVD ADER schemes. The task at hand is the computation of intermediate states $\mathbf{Q}_{i+1/2}$ in (6) or in (32)–(33) for use in (35). First we present a TVD version of the ADER scheme. Assume a MUSCL linear reconstruction as in (45). The problem is then to find an intermediate state $\mathbf{Q}_{i+1/2}$ corresponding to initial conditions of the form:

$$\mathbf{Q}(x,0) = \begin{cases} \mathbf{Q}_i(x) & \text{if } x < x_{i+1/2} \\ \mathbf{Q}_{i+1}(x) & \text{if } x > x_{i+1/2} \end{cases} \tag{51}$$

for Equations (1). A Taylor expansion about $t = 0$ at $x = x_{i+1/2}$ gives

$$\bar{\mathbf{Q}}_{i+1/2}(\tau) = \mathbf{Q}_{i+1/2}^{(0)} + \tau \partial_t \mathbf{Q} + O(\tau^2) \tag{52}$$

where $O(\tau^2)$ means that terms of second and higher order have been neglected. Here $\mathbf{Q}_{i+1/2}^{(0)}$ accounts for the *first instant* interaction of the linear states in (51) and may be computed in terms of the boundary extrapolated values given by (46), namely

$$\mathbf{Q}_{i+1/2}^{(0)} = \mathbf{Q}_{i+1/2}^{\text{PRICE-T}}(\mathbf{Q}_i^R, \mathbf{Q}_{i+1}^L) \tag{53}$$

Regarding the computation of the second term involving $\partial_t \mathbf{Q}$, use of (1) gives

$$\partial_t \mathbf{Q} = -\mathbf{A}_{i+1/2}^{(0)} \partial_x \mathbf{Q} \tag{54}$$

where $\mathbf{A}_{i+1/2}^{(0)} = \mathbf{A}(\mathbf{Q}_{i+1/2}^{(0)})$. Then (52) becomes

$$\mathbf{Q}_{i+1/2}(\tau) = \mathbf{Q}_{i+1/2}^{(0)} - \tau \mathbf{A}_{i+1/2}^{(0)} \partial_x \mathbf{Q} \tag{55}$$

where $\partial_x \mathbf{Q}$ remains to be computed. Then we define

$$\mathbf{V} \equiv \partial_x \mathbf{Q} \tag{56}$$

and note that $\mathbf{V}$ obeys the linear evolution system

$$\partial_t \mathbf{V} + \mathbf{A}_{i+1/2}^{(0)} \partial_x \mathbf{V} = \mathbf{0} \tag{57}$$

We pose and solve the derivative *Riemann problem* for (57) with initial condition

$$\mathbf{V}(x,0)=\begin{cases}\mathbf{V}_L\equiv(\partial_x\mathbf{Q})_i=\frac{\bar{\Delta}_i}{\Delta x}\\ \mathbf{V}_R\equiv(\partial_x\mathbf{Q})_{i+1}=\frac{\bar{\Delta}_{i+1}}{\Delta x}\end{cases} \tag{58}$$

This is *resolved* using the (PRICE-T) first-order monotone schemes of Section 2, namely

$$\mathbf{Q}^{(1)}_{i+1/2}=\mathbf{Q}^{\text{PRICE-T}}_{i+1/2}\left(\frac{\bar{\Delta}_i}{\Delta x},\frac{\bar{\Delta}_{i+1}}{\Delta x}\right) \tag{59}$$

Then (55) becomes

$$\mathbf{Q}_{i+1/2}(\tau)=\mathbf{Q}^{(0)}_{i+1/2}-\tau\mathbf{A}^{(0)}_{i+1/2}\mathbf{Q}^{(1)}_{i+1/2} \tag{60}$$

Finally, evaluation of the time-integral average of (60) between $t=0$ and $t=\Delta t$, as in (5), gives the intermediate state

$$\mathbf{Q}^{\text{ADER-TVD}}_{i+1/2}=\mathbf{Q}^{(0)}_{i+1/2}-\frac{1}{2}\Delta t\mathbf{A}^{(0)}_{i+1/2}\mathbf{Q}^{(1)}_{i+1/2} \tag{61}$$

Then (61) is used in the solution updating formula (35).

3.2.2. ENO version of ADER. A variant of scheme (61) may be obtained from the ENO approach to polynomial reconstruction, whereby uniform second-order accuracy of the scheme is maintained for smooth solutions. This feature of the resulting scheme would reduce the clipping of extrema produced by the TVD schemes.

As we are interested in second-order accurate schemes we only need a linear reconstruction of the data using a first-order polynomial

$$P_i(x)=\mathbf{Q}^n_i+S_i(x-x_i) \tag{62}$$

Following Harten and Osher [33], within a cell $I_i=[x_{i-1/2},x_{i+1/2}]$ there are three possible linear reconstructions characterized by three different slopes, which can be written as

$$S^1_i=\frac{\mathbf{Q}^n_{i+1}-\mathbf{Q}^n_i-\frac{1}{2}(\mathbf{Q}^n_{i+2}-2\mathbf{Q}^n_{i+1}+\mathbf{Q}^n_i)}{\Delta x} \tag{63}$$

$$S^2_i=\frac{\mathbf{Q}^n_{i+1}-\mathbf{Q}^n_i-\frac{1}{2}(\mathbf{Q}^n_{i+1}-2\mathbf{Q}^n_i+\mathbf{Q}^n_{i-1})}{\Delta x} \tag{64}$$

$$S^3_i=\frac{\mathbf{Q}^n_i-\mathbf{Q}^n_{i-1}+\frac{1}{2}(\mathbf{Q}^n_i-2\mathbf{Q}^n_{i-1}+\mathbf{Q}^n_{i-2})}{\Delta x} \tag{65}$$

In the ENO approach [22] one selects the polynomial with the smallest divided difference in absolute value. In the case of linear reconstruction this amounts to choosing the line with the smallest gradient or slope in absolute value, that is

$$S_i=\frac{1}{\Delta x}\,\text{minmod}[S^+_i,S^-_i] \tag{66}$$

where

$$S_i^+ = \mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n - \frac{1}{2}\,\mathrm{minmod}(\mathbf{Q}_{i+1}^n - 2\mathbf{Q}_i^n + \mathbf{Q}_{i-1}^n, \mathbf{Q}_{i+2}^n - 2\mathbf{Q}_{i+1}^n + \mathbf{Q}_i^n) \tag{67}$$

$$S_i^- = \mathbf{Q}_i^n - \mathbf{Q}_{i-1}^n + \frac{1}{2}\,\mathrm{minmod}(\mathbf{Q}_{i+1}^n - 2\mathbf{Q}_i^n + \mathbf{Q}_{i-1}^n, \mathbf{Q}_i^n - 2\mathbf{Q}_{i-1}^n + \mathbf{Q}_{i-2}^n) \tag{68}$$

and the minmod function chooses the smallest argument in absolute value if the arguments have the same sign and chooses zero otherwise.

Thus the boundary extrapolated values to be used in the ADER scheme are

$$\mathbf{Q}_i^R = \mathbf{Q}_i^n + \frac{1}{2}\,\Delta x S_i = P_i(x_{i+1/2}), \quad \mathbf{Q}_i^L = \mathbf{Q}_i^n - \frac{1}{2}\,\Delta x S_i = P_i(x_{i-1/2}) \tag{69}$$

Then we compute $\mathbf{Q}_{i+1/2}^{\text{ADER-ENO}}$ as in (61), with $\mathbf{Q}_{i+1/2}^{(0)}$ and $\mathbf{Q}_{i+1/2}^{(1)}$ computed as in (53) and (59), respectively. Finally, $\mathbf{Q}_{i+1/2}^{\text{ADER-ENO}}$ is used in the solution updating formula (35) if the PRICE-T scheme is adopted.

4. NUMERICAL RESULTS AND DISCUSSION

The numerical methods presented in this paper are generic and are applicable to any homogeneous system of hyperbolic equations. In this section we assess the performance the proposed methods using a model hyperbolic system written in primitive form, namely the time-dependent non-linear shallow water equations augmented by a passive scalar. The model system selected contains the essential features of hyperbolic systems to be encountered in practice, such as non-linear wave propagation, with smooth and discontinuous solutions. Moreover, for sufficiently simple initial conditions, such as those for a Riemann problem, the equations have exact solutions including rarefaction waves, contact discontinuities and shock waves. These exact solutions are invaluable in systematically assessing the performance of numerical methods intended for the solution of complex systems of equations of physical relevance. At this stage, we are assessing new numerical methods, we are not solving practical problems yet.

The equations written in non-conservative (primitive) form are

$$\partial_t \mathbf{Q} + \mathbf{A}(\mathbf{Q})\partial_x \mathbf{Q} = \mathbf{0} \tag{70}$$

where

$$\mathbf{Q} = \begin{bmatrix} h \\ u \\ \psi \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{bmatrix} \tag{71}$$

Here $h = h(x,t)$ is water depth, $u = u(x,t)$ is particle velocity, $\psi = \psi(x,t)$ is a passive scalar (e.g. pollutant concentration) and g (constant) is the acceleration due to gravity. The choice of primitive variables is not unique; other choices are also possible. Further details on the shallow water equations are found in Reference [4]. This hyperbolic system has real eigenvalues $\lambda_1 = u - a$, $\lambda_2 = u$ and $\lambda_3 = u + a$, where $a = \sqrt{gh}$ is the celerity. The eigenvalue $\lambda_2 = u$ corresponds to the contact wave associated with the passive scalar (concentration) ψ. These

equations are non-linear and contain smooth and discontinuous solutions, which pose challenges to any numerical method intended for solving realistic problems. We select two test problems with exact solutions. These exact solutions are used to carefully assess the performance of the methods presented in this paper.

Test 1 solves (70) in the domain $[0,50]$ with initial condition

$$\begin{aligned} &h(x,0)=h_L=1 \quad \text{if } x\leqslant 25 \quad \text{and} \quad h(x,0)=h_R=\tfrac{1}{2} \quad \text{if } x>25 \\ &u(x,0)=0 \quad \forall x\in[0,50] \\ &\psi(x,0)=\psi_L=1 \quad \text{if } x\leqslant 25 \quad \text{and} \quad \psi(x,0)=\psi_R=0 \quad \text{if } x>25 \end{aligned} \tag{72}$$

Computed and exact solutions are displayed at time $t=5$ s. The exact solution of this problem contains a left rarefaction, a right-facing shock wave and a contact discontinuity in the middle, across which the concentration ψ changes discontinuously. Rarefaction waves are smooth waves and numerical methods should be able to resolve these features accurately, especially their heads and tails, which contain discontinuities in spatial derivatives. Shock waves are discontinuous waves associated with the *genuinely non-linear fields* $u-a$ and $u+a$. These waves require (i) correct speed of propagation, (ii) sharp resolution of the transition zone and (iii) absence of spurious oscillations around the shock. Our primitive schemes are theoretically unable [15] to compute shocks with the correct propagation speed; however their sharpness and absence of spurious oscillations are achieved very satisfactorily. Contact waves are associated with the *linearly degenerate field* $\lambda_2=u$. The requirements for the computation of these waves are as for shocks, namely (i) to (iii) above. Our primitive centred schemes are able to meet all of the above requirements, with (ii) requiring qualification. In general, computation of waves associated with *linearly degenerate fields* is very challenging, even for modern numerical methods. One main difficulty is to preserve sharpness in the resolution of these waves in long-time evolution problems. Upwind methods are distinctly better than centred methods for these problems. In particular, good upwind methods resolve stationary, isolated contact waves exactly, while centred methods are here at their worst; our methods do not escape these observations. We note here that not all upwind-based methods resolve linear waves correctly; for example schemes based on flux vector splitting and those based on the HLL Riemann solver will behave like centred methods for linear fields, see Reference [3]. We emphasize that our primitive schemes can compute contact discontinuities with the correct propagation speed, as will be illustrated through the test problems.

Test 2 solves (70) in the domain $[0,50]$ with initial condition

$$\begin{aligned} &h(x,0)=1 \quad \forall\in[0,50] \\ &u(x,0)=u_L=-5 \quad \text{if } x\leqslant 25 \quad \text{and} \quad u(x,0)=u_R=5 \quad \text{if } x>25 \\ &\psi(x,0)=\psi_L=1 \quad \text{if } x\leqslant 25 \quad \text{and} \quad \psi(x,0)=\psi_R=0 \quad \text{if } x>25 \end{aligned} \tag{73}$$

Computed and exact solutions are displayed at time $t=2.5$s. The exact solution of this problem consists of two symmetric rarefaction waves and a stationary contact wave. Water depth and particle velocity are smooth throughout, while concentration changes discontinuously across the contact wave of zero speed of propagation. This test problem is challenging in two respects. First, many otherwise good numerical methods fail here because the strong rarefactions in the solution lead the schemes to compute negative water depths, which is obviously incorrect.

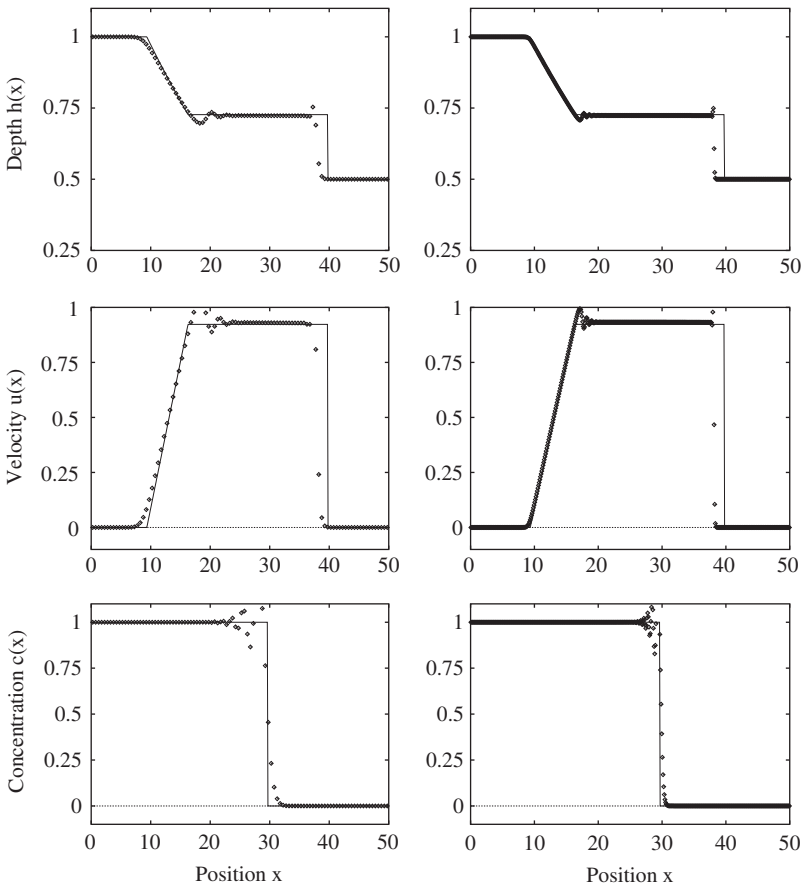

Figure 3. Test 1: PRICE-LW scheme (symbol) and exact solution (line). Results at time $t = 5.0$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

Codes *crash* when attempting to compute celerities. This problem is more severe for higher-order methods. The second difficulty is the correct resolution of the stationary contact wave. Our centred methods are not the best for this type of waves, but at least their speed of propagation is, on the average, correct and no spurious oscillations appear in the vicinity of the wave.

Numerical results are given in Figures 3–12. In all cases we have compared the numerical solution (symbols) with the exact solution (full line). In each figure the left column of results are obtained using a coarse mesh while the right column of results are obtained using a fine mesh. Figures 3–8 show the basic schemes constructed in Section 2 while Figures 9–12 show results for some of the second-order methods constructed in Section 3. The results shown in Figure 3 are obtained from scheme PRICE-LW, which is the non-conservative analogue of the two-step Lax–Wendroff method. The scheme is second order and linear, and therefore oscillatory. This is clearly seen by the oscillatory behaviour of the numerical solution in the

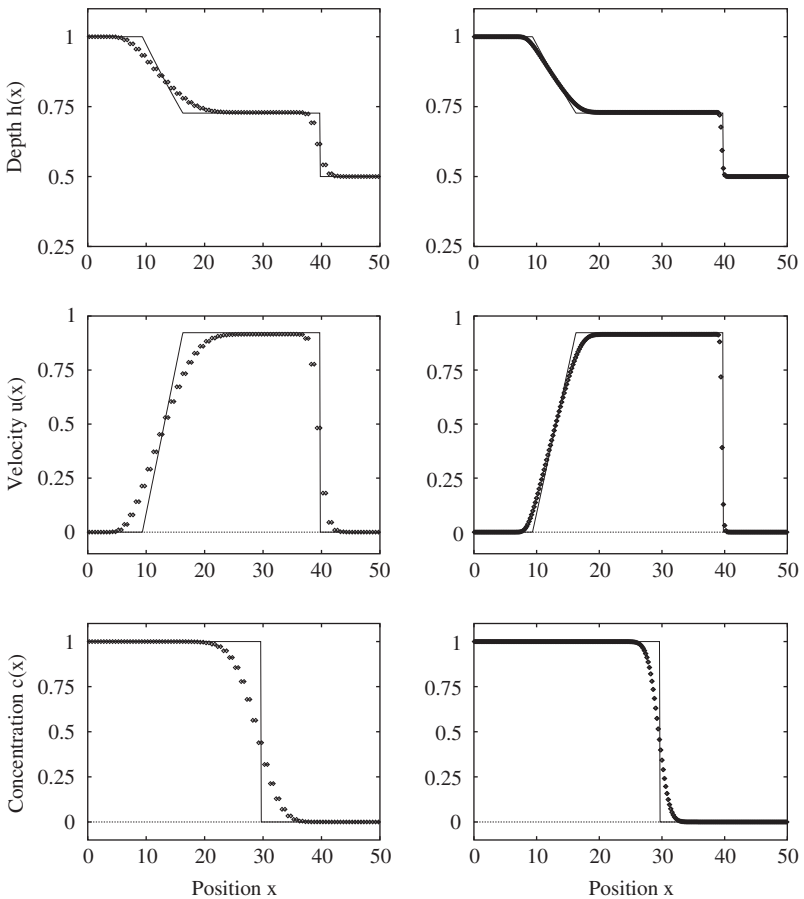

Figure 4. Test 1: PRICE-LF first-order scheme (symbol) and exact solution (line). Results at time $t = 5.0$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

vicinity of large gradients (shock, contact and also near the tail of the rarefaction). Also the position of the shock has a visible error; this is an expected feature of this non-conservative scheme. The fine mesh results in the right column of Figure 3 seem to indicate that the numerical solution converges to the wrong solution of the problem, see Reference [15] for a theoretical explanation.

The results of Figures 4–8 are all obtained from first-order monotone non-conservative schemes. Figure 4 shows the results obtained from PRICE-LF, the non-conservative analogue of the Lax–Friedrichs method. Three features stand out. The scheme shows *pairing* of neighbouring points, typical of the classical conservative Lax–Friedrichs scheme. The scheme is also very diffusive but monotone, again two features of the classical Lax–Friedrichs scheme. An interesting property of this non-conservative Lax–Friedrichs type scheme is that the position of the shock has only a small error. This is more clearly seen on the fine mesh results on the right-hand column. We also note that the position of the contact discontinuity seems

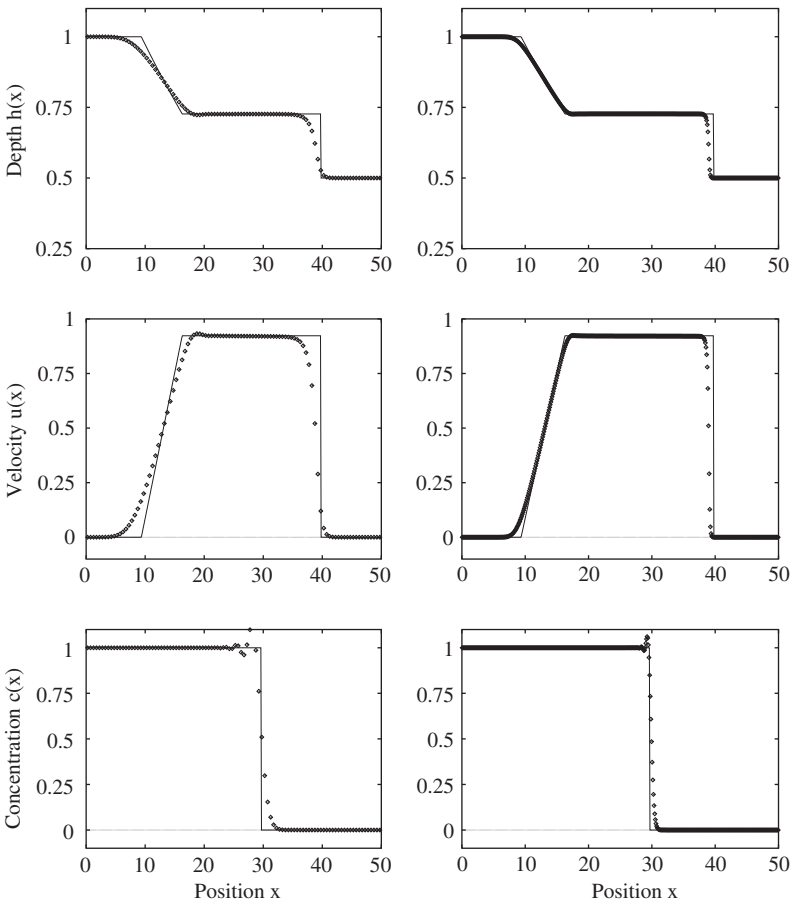

Figure 5. Test 1: PRICE-G first-order scheme (symbol) and exact solution (line). Results at time $t = 5.0$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.7$.

to be, on average, correct. This scheme could, in principle, form the basis for constructing high-order primitive centred methods. Figure 5 shows results for the PRICE-G scheme, the non-conservative analogue of the centred Godunov scheme. Note that for these results we have used a CFL number of 0.7, as this scheme has a more restrictive stability condition. As predicted, this first-order scheme is oscillatory; the oscillations are more visible for the slower waves. We note that the positional error of the shock wave is not large, but in general this scheme is not useful as it stands. High-order extensions could not be constructed, at least using current approaches, as these assume that the building block is a monotone scheme throughout its stability range. Figures 6–8 show results of three monotone first-order schemes with optimal stability range of CFL number unity. Figure 6 shows the results for the PRICE-T scheme which is a deterministic re-interpretation of the staggered grid random choice method. No spurious oscillations are seen and the results are distinctively more accurate than those of the primitive Lax–Friedrichs scheme PRICE-LF, shown in Figure 4. We also note that the

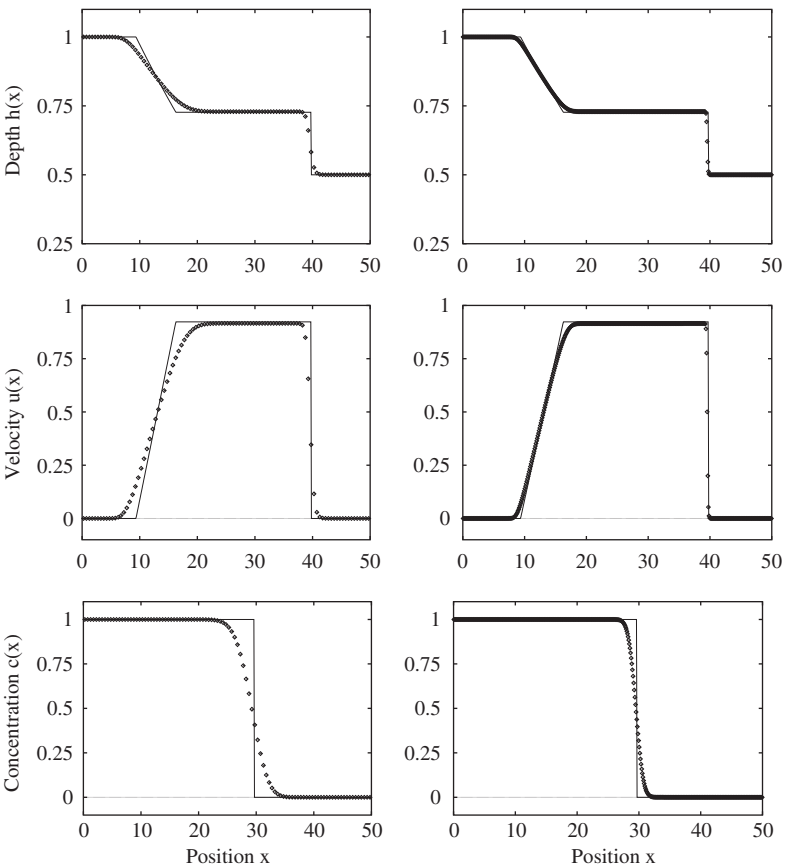

Figure 6. Test 1: PRICE-T first-order scheme (symbol) and exact solution (line). Results at time $t = 5.0$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

position of the shock has a very small error, certainly much smaller than that of PRICE-LW, Figure 3, and that of PRICE-G, Figure 5. This scheme appears to have the right properties for constructing high-order versions. Figures 7 and 8 shows the results obtained, respectively, with PRICE-F and PRICE-S, two more direct primitive analogues of the conservative FORCE scheme. Overall the results look very similar to those of PRICE-T, shown in Figure 6. As pointed out in Section 2 PRICE-T, PRICE-F and PRICE-S are identical for linear system with constant coefficients. We note that the positional error of the shock wave in the PRICE-F and PRICE-S schemes is larger than that of PRICE-T; the resolution of the other two waves is virtually identical in all three schemes. Based on the results of test 1 it seems as if PRICE-T is the best of all the schemes designed in Section 2. The closest competitors are PRICE-F and PRICE-S but these, in addition to the shock-positional error, have the disadvantage of being more complex and requiring more arithmetic operations.

We now discuss numerical results of some of the second-order methods constructed in Section 3. Figures 9 and 10 show results for the MUSCL-Hancock TVD extension of the

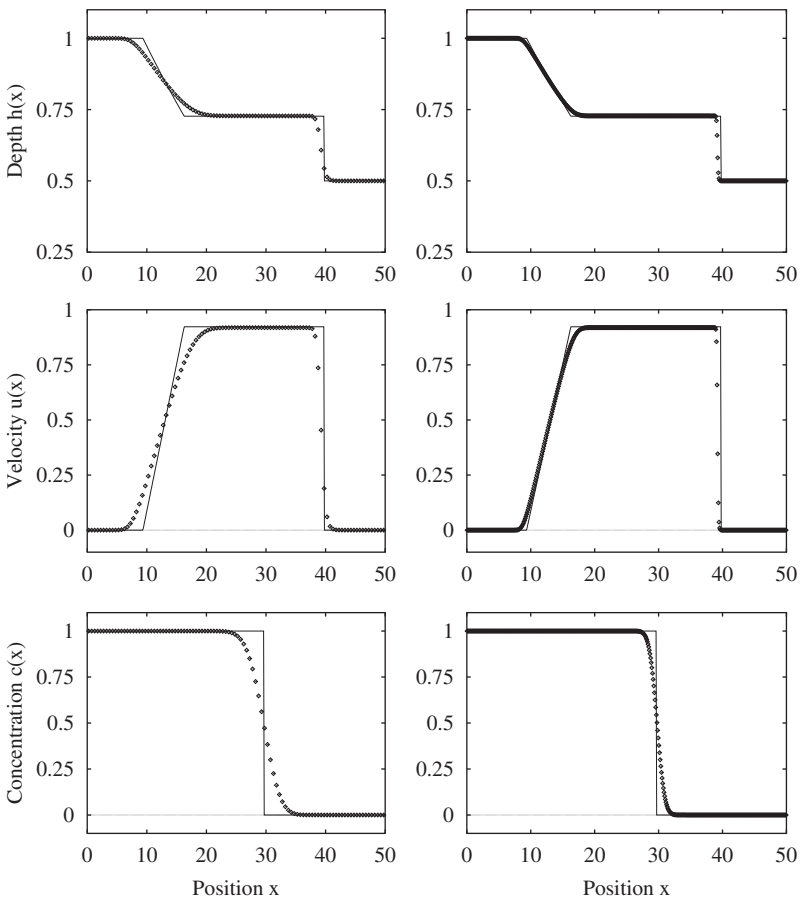

Figure 7. Test 1: PRICE-F first-order scheme (symbol) and exact solution (line). Results at time $t = 5.0$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

PRICE-T scheme with a superbee-like limiter (see Equation (A6) in Appendix A), as applied to tests 1 and 2. Figures 11 and 12 show results for the ADER extension of the PRICE-T scheme, with ENO reconstruction, for tests 1 and 2. The results of Figure 9 are indeed very satisfactory. There is good resolution of the smooth part of the flow and high resolution of discontinuities without spurious oscillations. The fine mesh results on the right column suggests that the scheme converges to a solution which is very close to the exact solution. The results of Figure 10, test 2, are generally satisfactory. Here the solution is smooth, except for the contact wave (concentration). No difficulties with depths near zero in the middle of the domain are experienced. However, the resolution of the tails of the two rarefactions (see the velocity plot) and that of the contact wave (see concentration plot) is not very satisfactory. Stationary discontinuities, as the one in this test problem, are the most difficult waves for centred (both primitive and conservative) schemes to resolve correctly; for these waves they have the largest dissipation, leading to spreading of the wave as time increases.

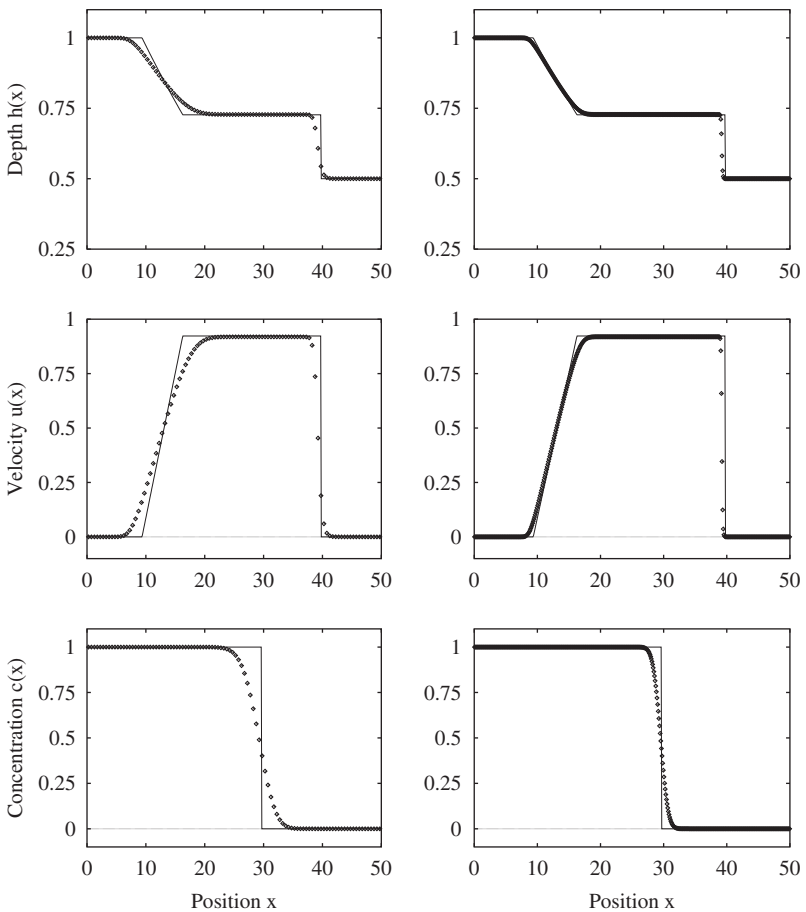

Figure 8. Test 1: PRICE-S first-order scheme (symbol) and exact solution (line). Results at time $t = 5.0$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

Good upwind methods, on the other hand, have a very clear advantage here, as they are able to resolve isolated stationary discontinuities exactly. The right column of Figure 9 shows fine mesh results. The scheme appears to converge to the correct solution, as the mesh is refined, although the behaviour of the scheme near the centre, under mesh refinement, is not totally satisfactory, convergence is very slow.

Figures 11 and 12 show the corresponding results for the ADER scheme with ENO reconstructions. These are to be compared with the respective results of Figures 9 and 10. Comments similar to those for the MUSCL-Hancock scheme apply. Generally, the two schemes, and also the ADER-TVD, give similar results, but there are some small differences. For example, for test 1, for both ADER schemes, the error in the position of the shock is slightly larger than that from the MUSCL-Hancock scheme, compare Figures 9 and 11. Also, the behaviour of the ADER scheme under mesh refinement for test 2 is more satisfactory, compare

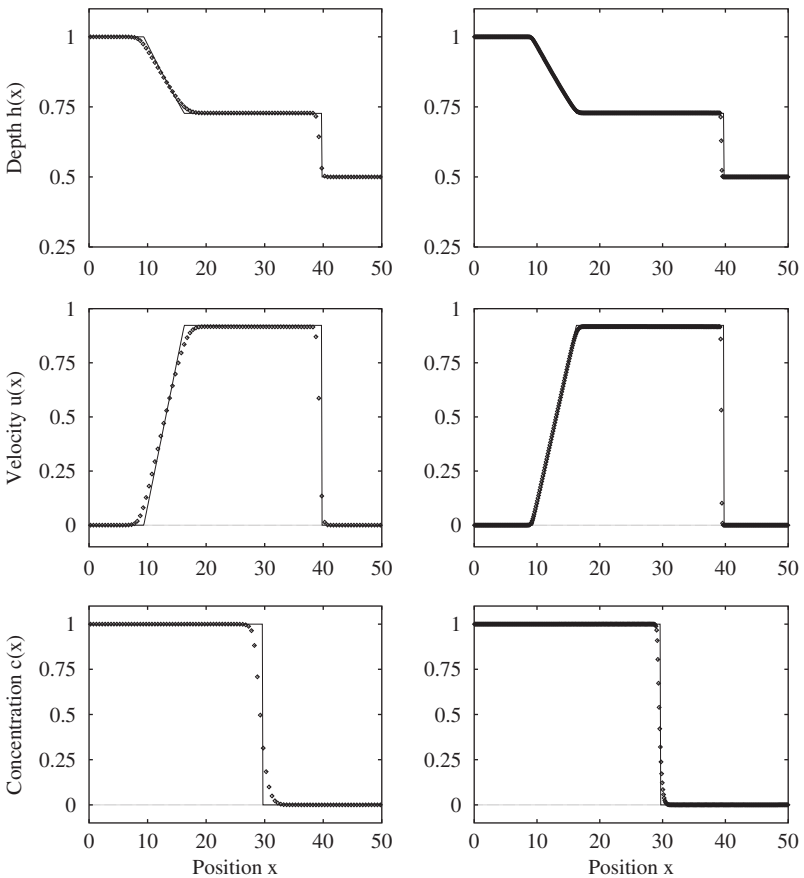

Figure 9. Test 1: MUSCL-Hancock TVD extension of the PRICE-T scheme (symbol) and exact solution (line). Results at time $t = 5.0$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

Figures 10 and 12 for example. We believe that the ADER approach has potential advantages, for example, when it comes to treating source terms. Also, there is a clear way of constructing ADER schemes of even higher order of accuracy in space and time.

5. SUMMARY, CONCLUSIONS AND FURTHER DEVELOPMENTS

We have constructed six new basic schemes that are primitive (non-conservative) and centred (non-upwind). PRICE-LW is a second-order accurate scheme, linear (and thus oscillatory); this scheme is the primitive analogue of the two-step Lax–Wendroff method. PRICE-LF is a primitive analogue of the classical conservative Lax–Fridrichs scheme. PRICE-G is a first-order non-monotone centred analogue of the conservative centred scheme of Godunov. PRICE-T is a first-order monotone scheme that is derived from a deterministic interpretation of the staggered grid version of the Random Choice Method. PRICE-T is one of three primitive

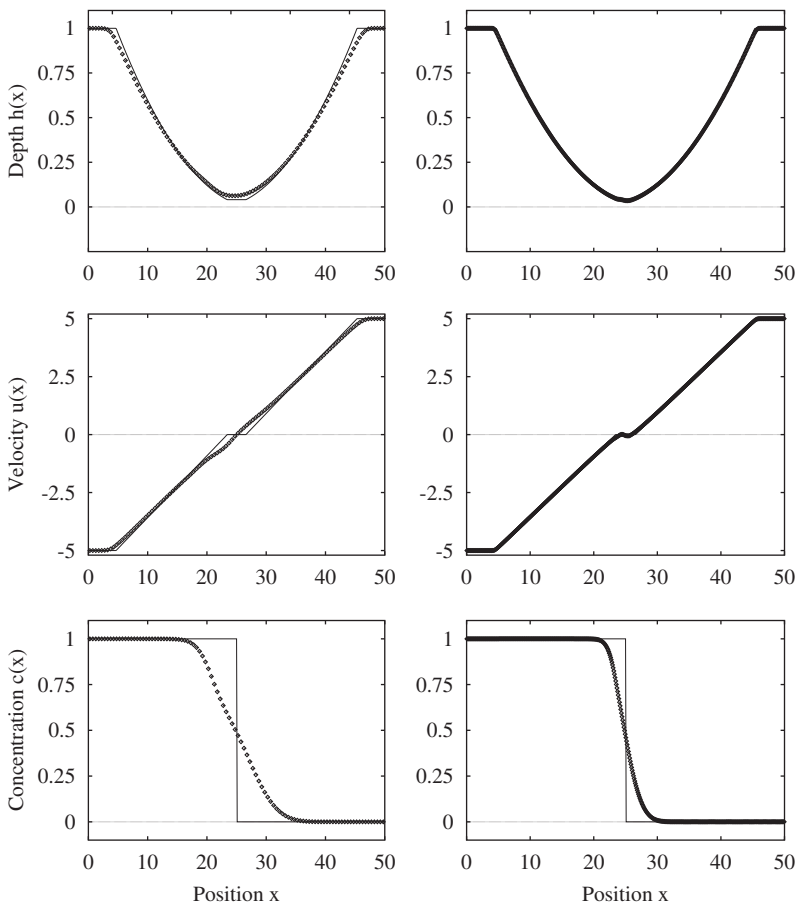

Figure 10. Test 2: MUSCL-Hancock TVD extension of the PRICE-T scheme (symbol) and exact solution (line). Results at time $t = 2.5$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

analogues of the conservative FORCE scheme of Toro. Finally, PRICE-F and PRICE-S are the other primitive analogues of FORCE; these schemes are first-order accurate and monotone. All of these basic schemes have optimal stability condition of CFL unity. The only exception to this is PRICE-G, which has a more restrictive stability condition. Useful schemes for high-order extensions are PRICE-LF, PRICE-T, and PRICE-S, with PRICE-T being perhaps the most attractive. We have extended some of the above schemes using two approaches originally developed for conservative methods, namely the MUSCL-Hancock approach and the ADER approach. In the ADER approach we have used two ways of dealing with linear reconstructions so as to avoid spurious oscillations: the ADER TVD scheme and ADER with ENO reconstruction. Extensive numerical experiments suggest that all the schemes are very satisfactory, with the ADER/ENO scheme being perhaps the most promising, first for dealing with source terms and secondly, because higher-order extensions are possible. Work currently in progress includes the application of some of these ideas to solve the mud flow

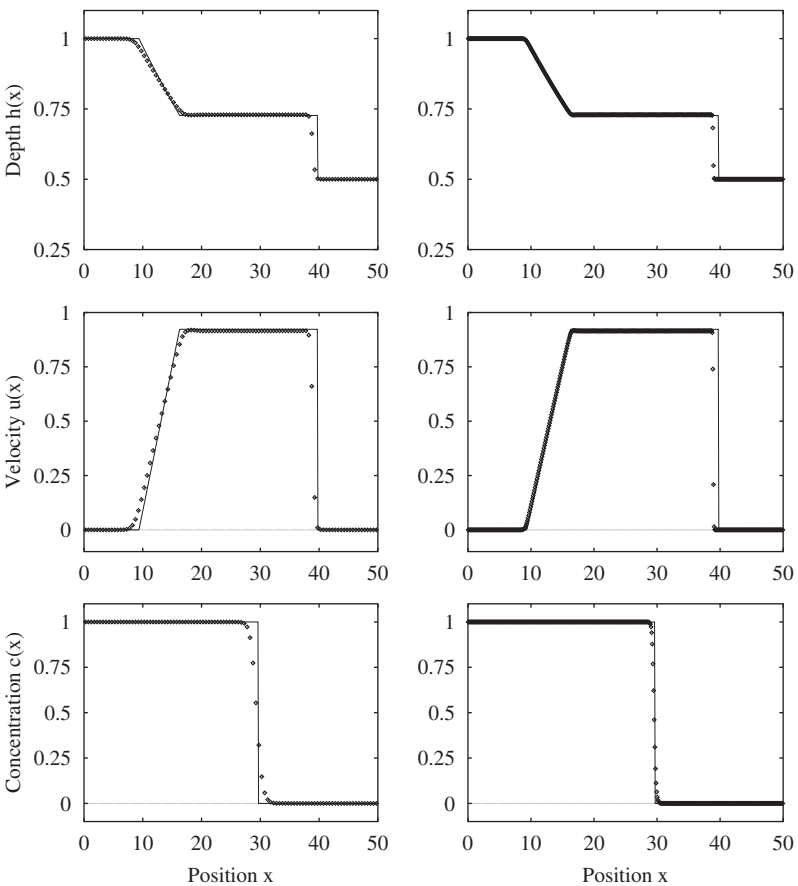

Figure 11. Test 1: ADER/ENO extension of the PRICE-T scheme (symbol) and exact solution (line). Results at time $t = 5.0$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

equations [8]. The schemes presented are generic and can be applied to any hyperbolic system in non-conservative form and for which solutions include smooth parts, contact discontinuities and weak shocks. The advantage of the schemes presented over upwind-based methods is simplicity and efficiency, and will be fully realized for hyperbolic systems in which the provision of upwind information is very costly or is not available.

APPENDIX A: TVD CONSTRAINTS AND SLOPE LIMITERS

Regarding the limited slopes (differences) $\bar{\Delta}_i$ in (45) we follow Section 14.4.3 of Reference [3] and write

$$\bar{\Delta}_i = \xi_i \Delta_i \tag{A1}$$

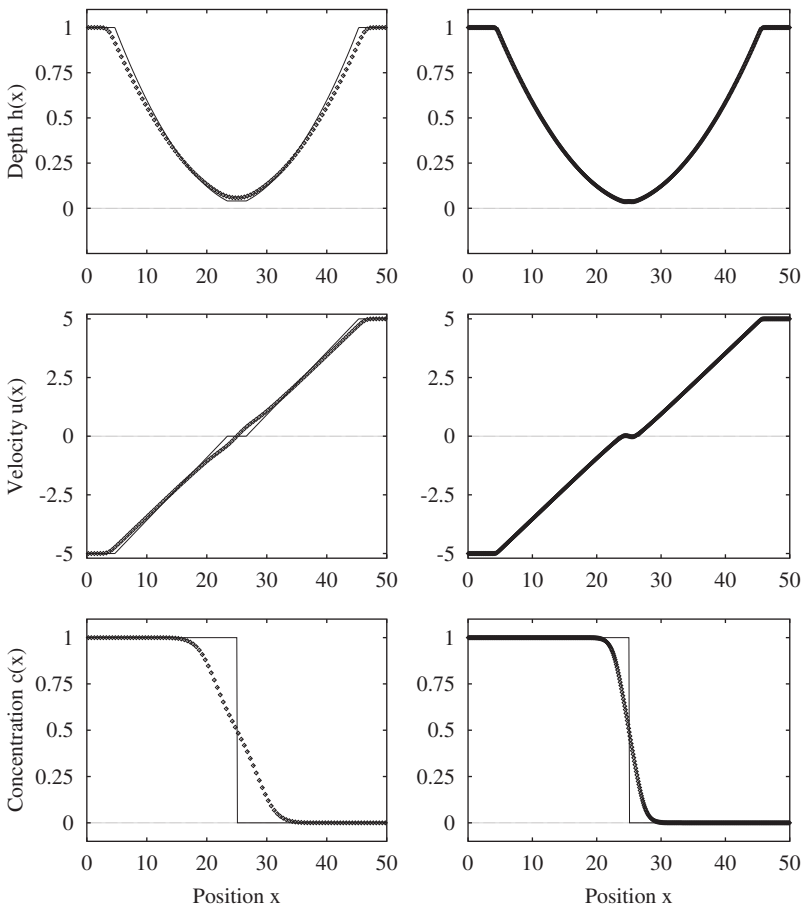

Figure 12. Test 2: ADER/ENO extension of the PRICE-T scheme (symbol) and exact solution (line). Results at time $t = 2.5$ s for mesh $M = 100$ (left) and $M = 500$ (right) and CFL $= 0.95$.

with Δ_i given, for example, as

$$\Delta_i = \frac{1}{2}(1+\omega)\Delta_{i-1/2} + \frac{1}{2}(1-\omega)\Delta_{i+1/2} \tag{A2}$$

with $\Delta_{i+1/2} = \mathbf{Q}^n_{i+1} - \mathbf{Q}^n_i$ and ω in the real interval $[-1, 1]$. This approach leads to a TVD region for $\xi(r)$ given as follows:

$$\xi(r) = 0 \quad \text{for} \quad r \leqslant 0, \quad 0 \leqslant \xi(r) \leqslant \min\{\xi_L(r), \xi_R(r)\} \quad \text{for } r > 0 \tag{A3}$$

where

$$\xi_L(r) = \frac{2\beta_{i-1/2} r}{1-\omega+(1+\omega)r}$$
$$\xi_R(r) = \frac{2\beta_{i+1/2}}{1-\omega+(1+\omega)r} \tag{A4}$$
$$r = \frac{\Delta_{i-1/2}}{\Delta_{i+1/2}}$$

and

$$\beta_{i-1/2} = \frac{2}{1+c}, \quad \beta_{i+1/2} = \frac{2}{1-c} \tag{A5}$$

The coefficients $\beta_{i-1/2}$ and $\beta_{i+1/2}$ are in general functions of the Courant number c for the single wave present in the scalar case, for which the TVD condition has been derived. For convenience we eliminate this dependency by taking their limiting values $\beta_{i-1/2} = \beta_{i+1/2} = 1$ in (A5). Slope limiters that are analogous to conventional flux limiters, such as SUPERBEE and MINBEE were constructed in Reference [3]. We stress however that they are only analogous, not equivalent. A slope limiter that is analogous to the SUPERBEE flux limiter is

$$\xi_{\text{sb}}(r) = \begin{cases} 0, & \text{if } r \leqslant 0 \\ 2r, & \text{if } 0 \leqslant r \leqslant \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leqslant r \leqslant 1 \\ \min\{r, \xi_R(r), 2\}, & \text{if } r \geqslant 1 \end{cases} \tag{A6}$$

A van Leer-type slope limiter is

$$\xi_{\text{vl}}(r) = \begin{cases} 0, & \text{if } r \leqslant 0 \\ \min\{\frac{2r}{1+r}, \xi_R(r)\}, & \text{if } r \geqslant 0 \end{cases} \tag{A7}$$

A van Albada-type slope limiter is

$$\xi_{\text{va}}(r) = \begin{cases} 0, & \text{if } r \leqslant 0 \\ \min\{\frac{r(1+r)}{1+r^2}, \xi_R(r)\}, & \text{if } r \geqslant 0 \end{cases} \tag{A8}$$

A MINBEE-type slope limiter is

$$\xi_{\text{mb}}(r) = \begin{cases} 0, & \text{if } r \leqslant 0 \\ r, & \text{if } 0 \leqslant r \leqslant 1 \\ \min\{1, \xi_R(r)\}, & \text{if } r \geqslant 1 \end{cases} \tag{A9}$$

In practice we recommend (A6) and (A7). Choices (A8) and (A9) are more diffusive.

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